

POSITIVE SIMPLICIAL VOLUME IMPLIES VIRTUALLY POSITIVE SEIFERT VOLUME FOR 3-MANIFOLDS

PIERRE DERBEZ, YI LIU, HONGBIN SUN, AND SHICHENG WANG

ABSTRACT. In this paper, it is shown that for any closed orientable 3-manifold with positive simplicial volume, the growth of the Seifert volume of its finite covers is faster than the linear rate. In particular, each closed orientable 3-manifold with positive simplicial volume has virtually positive Seifert volume. The result reveals certain fundamental difference between the representation volume of the hyperbolic type and the Seifert type. The proof is based on developments and reactions of recent results on virtual domination and on virtual representation volumes of 3-manifolds.

CONTENTS

1. Introduction	1
Acknowledgement	5
2. Preliminaries	5
2.1. Geometry and topology of 3-manifolds after Thurston	5
2.2. Representation volumes of closed manifolds	7
3. Positive simplicial volume implies virtually positive Seifert volume	8
3.1. Virtual representation through geometric decomposition	8
3.2. Proof of Theorem 1.6	9
4. Efficient virtual domination by hyperbolic 3-manifolds	12
4.1. URC manifolds	13
4.2. Virtual domination through URC 3-manifolds	13
4.3. Virtual Seifert volume of closed hyperbolic 3-manifolds	16
5. Positive simplicial volume implies unbounded virtual Seifert volume	16
5.1. Proof of Theorem 1.8	17
5.2. CI completions of hyperbolic 3-manifolds	18
5.3. Virtual representations through CI completions	20
6. On covering invariants	23
References	25

1. INTRODUCTION

The representation volume of 3-manifolds is a beautiful theory, exhibiting rich connections with many branches of mathematics. The behavior of those volume functions appears to be quite mysterious; for example, their values are hard to predict except in very few nice cases. On the other hand, for most motivating

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applications, it suffices to estimate the growth of such volumes for finite covers of the considered 3-manifold. In this paper, we are intended to investigate the possibility of the latter, which is as interesting as a topic on its own right.

To be more specific, let us introduce some basic notations and mention some known properties of the representation volume. Let G be either

$$\text{Iso}_+\mathbf{H}^3 \cong \text{PSL}(2; \mathbb{C}),$$

the orientation-preserving isometry group of the 3-dimensional hyperbolic geometry, or

$$\text{Iso}_e\widetilde{\text{SL}_2(\mathbb{R})} \cong \mathbb{R} \times_{\mathbb{Z}} \widetilde{\text{SL}_2(\mathbb{R})},$$

the identity component of the isometry group of the Seifert geometry. For any closed orientable 3-manifold N and any representation $\rho: \pi_1 N \rightarrow G$, denote by $\text{vol}_G(N, \rho)$ the (unsigned) volume of ρ . We denote the set of G -representation volumes of N by

$$\text{vol}(N, G) = \{\text{vol}_G(N, \rho) : \rho \text{ any representation } \pi_1 N \rightarrow G\},$$

which is a subset of the interval $[0, +\infty)$.

The following theorem contains a collection of fundamental facts in the theory of representation volumes [BG1, Re].

Theorem 1.1. *Let N be a closed orientable 3-manifold.*

- (1) *The sets of values $\text{vol}(N, \text{Iso}_+\mathbf{H}^3)$ and $\text{vol}(N, \text{Iso}_e\widetilde{\text{SL}_2(\mathbb{R})})$ are both finite. Hence the following values*

$$\text{HV}(N) = \max \text{vol}(N, \text{Iso}_+\mathbf{H}^3)$$

and

$$\text{SV}(N) = \max \text{vol}(N, \text{Iso}_e\widetilde{\text{SL}_2(\mathbb{R})})$$

exist in $[0, +\infty)$ depending only on N .

- (2) *If N admits a hyperbolic geometric structure, then $\text{HV}(N)$ equals the usual hyperbolic volume of N , reached by any discrete and faithful representation. A similar statement holds for $\text{SV}(N)$ when N admits a Seifert geometric structure.*
- (3) *If P_1, \dots, P_s are the prime factors of N in the Kneser–Milnor decomposition, then*

$$\text{HV}(N) = \text{HV}(P_1) + \dots + \text{HV}(P_s).$$

A similar formula holds for $\text{SV}(N)$.

- (4) *For any map $f: M \rightarrow N$ between closed orientable 3-manifolds,*

$$\text{HV}(M) \geq |\deg f| \cdot \text{HV}(N).$$

The same comparison holds for $\text{SV}(M)$ and $\text{SV}(N)$.

The values $\text{HV}(N)$ and $\text{SV}(N)$ in the conclusion of Theorem 1.1 (1) are called the *hyperbolic volume* and the *Seifert volume* of N , respectively. In light of Theorem 1.1 (3), we assume from now on that all the closed orientable 3-manifolds considered are prime, unless specified otherwise. This is especially convenient when we speak of the geometric decomposition of the 3-manifold.

Remark 1.2. Representation volumes are introduced and studied by R. Brooks and W. Goldman [BG1, BG2] as a generalization of the simplicial volume originally due to M. Gromov [Gr]. Among the eight 3-dimensional geometries of W. P. Thurston, \mathbf{H}^3 and $\mathrm{SL}_2(\mathbb{R})$ are the only two that yield nontrivial invariants, the hyperbolic volume and the Seifert volume, accordingly. Recall that the *simplicial volume* of a closed orientable 3-manifold N is defined to be $v_3\|N\|$, the product of the Gromov norm $\|N\|$ of N and the volume v_3 of the ideal regular hyperbolic tetrahedron; it is known to equal to the sum of the classical hyperbolic volume of the hyperbolic pieces [So].

Like the simplicial volume, the volumes of Brooks–Goldman satisfy the *domination property*, as stated by Theorem 1.1 (4). It follows that if either of the volumes $\mathrm{HV}(N)$ or $\mathrm{SV}(N)$ is positive, then the set of mapping degrees $D(M, N)$ of N by any given 3-manifold M must be finite. Unlike the simplicial volume, neither the hyperbolic volume nor the Seifert volume satisfies the *covering property*, [DLW, Corollary 1.8]; see Section 6 for some further discussion.

It can be inferred from Theorem 1.1 and Remark 1.2 that non-vanishing $\mathrm{HV}(N)$ or $\mathrm{SV}(N)$ contains interesting information about the topology of the 3-manifold N . However, such information seems difficult to characterize. For example, the vanishing or non-vanishing of $\mathrm{SV}(N)$ implies nothing about the behavior of $\mathrm{HV}(N)$, [BG1, Section 4 and 5]; and except for the geometric case (Theorem 1.1 (2)), the geometry of pieces fails to detect the vanishing or non-vanishing of $\mathrm{HV}(N)$ or $\mathrm{SV}(N)$ either, [DLW, Theorem 1.7]. On the other hand, the existence of some finite cover of N with non-vanishing representation volume turns out to be a question more accessible. An affirmative answer would be practically useful: It implies the finiteness of the set of mapping degrees as before. Motivated by that application, it has been discovered that any non-geometric graph manifold admits a finite cover of positive Seifert volume, [DW1, DW2]; a much more general construction that invokes Chern–Simons theoretic calculations and virtual properties of 3-manifolds shows that a right geometric piece implies virtually positive volume of the right geometry [DLW, Theorems 1.6]:

Theorem 1.3. *Suppose that N is a closed orientable non-geometric prime 3-manifold.*

- (1) *If N contains at least one hyperbolic geometric piece, then the hyperbolic volume of some finite cover of N is positive.*
- (2) *If N contains at least one Seifert geometric piece, then the Seifert volume of some finite cover of N is positive.*

Despite the seeming parallelism so far, the hyperbolic volume and the Seifert volume behave drastically differently with respect to finite covers. In this paper, we support this point by investigating two problems proposed in [DLW, Section 8]:

Problem 1.4. Estimate the growth of virtual hyperbolic volume and virtual Seifert volume.

Problem 1.5. Is the Seifert volume of a closed prime 3-manifold virtually positive if it has positive simplicial volume?

The main results of this paper address Problem 1.5 affirmatively (Theorem 1.6) and Problem 1.4 partially for 3-manifolds of positive simplicial volume (Theorem

1.8 and Remark 1.9), showing that the growth of virtual Seifert volume is super-linear while the growth of virtual hyperbolic volume is linear. On Problem 1.5, the case of closed hyperbolic 3-manifolds is already known as a direct consequence of the much stronger virtual domination theorem [Su] (quoted as Theorem 1.10 below); so essentially it remains to treat the case of non-geometric 3-manifolds (with only hyperbolic pieces). On Problem 1.4, it is easy to observe that the growth of virtual Seifert volume for a closed Seifert geometric 3-manifold is linear, indeed, in a constant rate equal to its Seifert volume. Comparing with our result, we are left with the impression that the growth of virtual hyperbolic volume might be largely governed by the simplicial volume, and the growth of virtual Seifert volume appears to be more sensitive to the geometric decomposition.

The main results of this paper are stated as the following Theorems 1.6 and 1.8:

Theorem 1.6. *If M is a closed orientable 3-manifold with positive simplicial volume, then there is a finite cover \tilde{M} of M with positive Seifert volume.*

Combining with results of [DW2, DLW, DSW], we infer immediately the following characterization:

Corollary 1.7. *Suppose that N is a closed orientable 3-manifold. Then the following three statements are equivalent.*

- (1) *The set of mapping degrees $D(M, N)$ is finite for every closed orientable 3-manifold M .*
- (2) *The Seifert volume of some finite cover of N is positive.*
- (3) *At least one prime factor of N is Seifert geometric, or hyperbolic, or non-geometric.*

Theorem 1.8. *For any closed oriented 3-manifold M with non-vanishing simplicial volume, the set of values*

$$\left\{ \frac{\text{SV}(M')}{[M' : M]} : M' \text{ any finite cover of } M \right\}$$

has no upper bound in $[0, +\infty)$.

Remark 1.9. In contrast, it is evident by [Re, Theorem B] and Theorem 1.1 that the set of values

$$\left\{ \frac{\text{HV}(M')}{[M' : M]} : M' \text{ any finite cover of } M \right\}$$

is contained by the interval $[0, v_3 \|M\|]$ where the upper bound is the simplicial volume of M .

Theorem 1.8 is significantly stronger than Theorem 1.6. Let us take a closer look at the geometric case to illustrate their difference in the proof. As mentioned, when M is assumed to be geometric, hence hyperbolic, Theorem 1.6 is implied by the following result of [Su], by taking N to be a target with positive Seifert volume:

Theorem 1.10. *For any closed oriented hyperbolic 3-manifold M , and any closed oriented 3-manifold N , M admits a finite cover \tilde{M} , such that there exists a π_1 -surjective degree-2 map $f : \tilde{M} \rightarrow N$.*

Even though Theorem 1.10 is a powerful construction, employing deep theories including [KM, LM] and [Ag, Wi] on building and separating certain quasiconvex subgroups in closed hyperbolic 3-manifold groups, the construction provides no

control on the degree $[\tilde{M} : M]$. So Theorem 1.8 stays beyond the reach of Theorem 1.10. Armed with a more recent result of A. Gaifullin [Ga], we prove the following Theorem 1.11 based on Theorem 1.10. The improved construction is supplied with a desired efficient control of the mapping degree:

Theorem 1.11. *For any closed oriented hyperbolic 3-manifold M , there exists a positive constant $c(M)$ such that the following statement holds. For any closed oriented 3-manifold N and any $\epsilon > 0$, there exists a finite cover M' of M which admits a non-zero degree map $f : M' \rightarrow N$, such that*

$$\|M'\| \leq c(M) \cdot |\deg(f)| \cdot (\|N\| + \epsilon).$$

To prove Theorems 1.6 and 1.8 in the non-geometric case, it is tempting to extend Theorems 1.10 and 1.11 to mixed 3-manifolds, but we do not have available tools for that project. Instead, we follow the framework of [DLW] and integrate the virtual domination theorems. The reaction between Theorem 1.10 and the fundamental construction for Theorem 1.3 is fairly direct and illustrating, so we present it and prove Theorem 1.6 as a warm-up. The proof of Theorem 1.8 is relatively more sophisticated, not only because of Theorem 1.11, but requiring some details of [DLW]. In particular, we introduce an auxiliary notion called *CI completions* to formalize some useful idea underlying the construction of [DLW], (see Subsection 5.2).

All the arguments are based on explicitly stated results, and the exposition is kept otherwise self-contained. The organization of this paper is as the following. The proofs of Theorem 1.6, and Theorem 1.11, and Theorem 1.8 occupy Sections 3, 4, and 5, respectively. Section 2 includes preliminaries on 3-manifold topology and representation volume. Section 6 contains some further questions and observations.

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2. PRELIMINARIES

In this section, we review the geometric decomposition of 3-manifolds and the theory of representation volumes.

2.1. Geometry and topology of 3-manifolds after Thurston. Let N be a connected compact prime orientable 3-manifold with toral or empty boundary. As a consequence of the geometrization of 3-manifolds [Th1, Th2] achieved by G. Perelman and W. Thurston, exactly one of the following case holds:

- Either N is geometric, supporting one of the following eight geometries: \mathbf{H}^3 , $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, $\mathbf{H}^2 \times \mathbb{R}$, Sol, Nil, \mathbb{R}^3 , \mathbf{S}^3 and $\mathbf{S}^2 \times \mathbb{R}$ (where \mathbf{H}^n , \mathbb{R}^n and \mathbf{S}^n are the n -dimensional hyperbolic space, Euclidean space, and spherical space respectively);
- or N has a canonical nontrivial geometric decomposition. In other words, there is a nonempty minimal union $\mathcal{T}_N \subset N$ of disjoint essential tori and Klein bottles in N , unique up to isotopy, such that each component of

$N \setminus \mathcal{T}_N$ is either Seifert fibered or atoroidal. In the Seifert fibered case, the piece supports the $\mathbf{H}^2 \times \mathbb{R}$ geometry and the $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ geometry. In the atoroidal case, the piece supports the \mathbf{H}^3 geometry.

When N has nontrivial geometric decomposition, we call the components of $N \setminus \mathcal{T}_N$ the *geometric pieces* of N , or more specifically, *Seifert pieces* or *hyperbolic pieces* according to their geometry.

Traditionally, there is another decomposition introduced by Jaco–Shalen [JS] and Johannson [Joh], known as the *JSJ decomposition*. When N contains no essential Klein bottles and has a nontrivial geometric decomposition, the JSJ decomposition of N coincides with its geometric decomposition, so the cutting tori and the geometric pieces are also the *JSJ tori* and the *JSJ pieces*, respectively. Possibly after passing to a double cover of N , we may assume that N contains no essential Klein bottle.

A hyperbolic piece J can be realized as a complete hyperbolic 3-manifold of finite volume, unique up to isometry (by Mostow Rigidity). Let J be a compact, orientable 3-manifold whose boundary consists of tori T_1, \dots, T_p and whose interior admits a complete hyperbolic metric. Identify J with the complement of p disjoint cusps in the corresponding hyperbolic manifold, then ∂J has a Euclidean metric induced from the hyperbolic structure, and each closed Euclidean geodesic in ∂J has the induced length. The Hyperbolic Dehn Filling Theorem of Thurston [Th1, Theorem 5.8.2] can be stated in the following form.

Theorem 2.1. *There is a positive constant C such that the closed 3-manifold $J(\zeta_1, \dots, \zeta_n)$ obtained by Dehn filling each T_i along a slope $\zeta_i \subset T_i$ admits a complete hyperbolic structure if each ζ_i has length greater than C . Moreover, with suitably chosen base points, $J(\zeta_1, \dots, \zeta_n)$ converges to the corresponding cusped hyperbolic 3-manifold in the Gromov–Hausdorff sense as the minimal length of ζ_i tends to infinity.*

A Seifert piece J of a non-geometric prime closed 3-manifold N supports both the $\mathbf{H}^2 \times \mathbb{R}$ geometry and the $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ geometry. In this paper, we are more interested in the latter case, so we describe the structure of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ geometric manifolds in the following.

We consider the group $\mathrm{PSL}(2; \mathbb{R})$ as the orientation preserving isometries of the hyperbolic 2-space $\mathbf{H}^2 = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ with i as a base point. In this way $\mathrm{PSL}(2; \mathbb{R})$ is a (topologically trivial) circle bundle over \mathbf{H}^2 . Let $p: \widetilde{\mathrm{SL}_2(\mathbb{R})} \rightarrow \mathrm{PSL}(2; \mathbb{R})$ be the universal covering of $\mathrm{PSL}(2; \mathbb{R})$ with the induced metric, then $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ is a line bundle over \mathbf{H}^2 . For any $\alpha \in \mathbb{R}$, denote by $\mathrm{sh}(\alpha)$ the element of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ whose projection into $\mathrm{PSL}(2; \mathbb{R})$ is given by $\begin{pmatrix} \cos(2\pi\alpha) & \sin(2\pi\alpha) \\ -\sin(2\pi\alpha) & \cos(2\pi\alpha) \end{pmatrix}$.

Then the set $\{\mathrm{sh}(n) \mid n \in \mathbb{Z}\}$ is the kernel of p , as well as the center of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, acting by integral translation along the fibers of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$. By extending this \mathbb{Z} -action on the fibers by the \mathbb{R} -action, we get the whole identity component of the isometry group

of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$. To summarize, we have the following diagram of central extensions

$$\begin{array}{ccccccccc} \{0\} & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{\mathrm{SL}_2(\mathbb{R})} & \longrightarrow & \mathrm{PSL}(2; \mathbb{R}) & \longrightarrow & \{1\} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathrm{Iso}_e \widetilde{\mathrm{SL}_2(\mathbb{R})} & \longrightarrow & \mathrm{PSL}(2; \mathbb{R}) & \longrightarrow & \{1\} \end{array}$$

In particular, the group $\mathrm{Iso}_e \widetilde{\mathrm{SL}_2(\mathbb{R})}$ is generated by $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ and the image of \mathbb{R} which intersect with each other in the image of \mathbb{Z} . More precisely, we state the following useful lemma which is easy to check.

Lemma 2.2. *We have the identification $\mathrm{Iso}_e \widetilde{\mathrm{SL}_2(\mathbb{R})} = \mathbb{R} \times_{\mathbb{Z}} \widetilde{\mathrm{SL}_2(\mathbb{R})}$: where $(x, h) \sim (x', h')$ if and only if there exists an integer $n \in \mathbb{Z}$ such that $x' - x = n$ and $h' = \mathrm{sh}(-n) \circ h$.*

From [BG2] we know that a closed orientable 3-manifold J supports the $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ geometry, i.e. there is a discrete and faithful representation $\psi : \pi_1 J \rightarrow \mathrm{Iso} \widetilde{\mathrm{SL}_2(\mathbb{R})}$, if and only if J is a Seifert fibered space with non-zero Euler number $e(J)$ and the base orbifold $\chi_{O(J)}$ has negative Euler characteristic.

2.2. Representation volumes of closed manifolds. In this subsection, we recall the definition of volume of representations. There are a few equivalent definitions, and we will only state one of them.

Given a semi-simple, connected Lie group G and a closed oriented manifold M^n of the same dimension as the contractible space $X^n = G/K$, where K is a maximal compact subgroup of G . We can associate to each representation $\rho : \pi_1 M \rightarrow G$, a volume $\mathrm{vol}_G(M, \rho)$ in the following way.

First fix a G -invariant Riemannian metric g_X on X , and denote by ω_X the corresponding G -invariant volume form. Let \widetilde{M} denote the universal covering of M . We think of the elements \tilde{x} of \widetilde{M} as the homotopy classes of paths $\gamma : [0, 1] \rightarrow M$ with $\gamma(0) = x_0$ which are acted by $\pi_1(M, x_0)$ by setting $[\sigma] \cdot \tilde{x} = [\sigma \cdot \gamma]$, where \cdot denotes the composition of paths.

A developing map $D_\rho : \widetilde{M} \rightarrow X$ associated to ρ is a $\pi_1 M$ -equivariant map such that for any $x \in \widetilde{M}$ and $\alpha \in \pi_1 M$, we have

$$D_\rho(\alpha \cdot x) = \rho(\alpha) D_\rho(x)$$

where $\rho(\alpha)$ acts on X as an isometry. Such a map does exist and can be constructed explicitly as in [BCG]: Fix a triangulation Δ_M of M , then it lifts to a triangulation $\Delta_{\widetilde{M}}$ of \widetilde{M} , which is $\pi_1 M$ -invariant. Then fix a fundamental domain Ω of M in \widetilde{M} such that the zero skeleton $\Delta_{\widetilde{M}}^0$ misses the frontier of Ω . Let $\{x_1, \dots, x_l\}$ be the vertices of $\Delta_{\widetilde{M}}^0$ in Ω , and let $\{y_1, \dots, y_l\}$ be any l points in X . We first set

$$D_\rho(x_i) = y_i, \quad i = 1, \dots, l.$$

Then extend D_ρ in a $\pi_1 M$ -equivariant way to $\Delta_{\widetilde{M}}^0$: for any vertex x in $\Delta_{\widetilde{M}}^0$, there is a unique vertex x_i in Ω and $\alpha_x \in \pi_1 M$ such that $\alpha_x \cdot x_i = x$, and we set $D_\rho(x) = \rho(\alpha_x)^{-1} D_\rho(x_i)$. Finally we extend D_ρ to edges, faces, etc., and n -simplices of $\Delta_{\widetilde{M}}$ by straightening their images to totally geodesic objects using the homogeneous metric on the contractible space X . This map is unique up to equivariant homotopy.

Then $D_\rho^*(\omega_X)$ is a $\pi_1 M$ -invariant closed n -form on \widetilde{M} , therefore can be thought of as a closed n -form on M . Then we define

$$\text{vol}_G(M, \rho) = \int_M D_\rho^*(\omega_X) = \sum_{i=1}^s \epsilon_i \text{vol}_X(D_\rho(\widetilde{\Delta}_i))$$

Here $\{\Delta_1, \dots, \Delta_s\}$ are the n -simplices of Δ_M , $\widetilde{\Delta}_i$ is a lift of Δ_i and $\epsilon_i = \pm 1$ depends on whether $D_\rho|_{\widetilde{\Delta}_i}$ preserves the orientation or not.

3. POSITIVE SIMPLICIAL VOLUME IMPLIES VIRTUALLY POSITIVE SEIFERT VOLUME

In this section, we adapt Theorem 1.10 of [Su] to the framework of [DLW] to prove Theorem 1.6.

3.1. Virtual representation through geometric decomposition. We recall some results from [DLW]. The following additivity principle allows us to compute the representation volume by information on the JSJ pieces. It is proved by using the relation between the representation volume and the Chern–Simons theory.

Theorem 3.1 (Additivity principle, [DLW, Theorem 3.5]; see also [DW2]). *Let M be an oriented closed 3-manifold with JSJ tori T_1, \dots, T_r and JSJ pieces J_1, \dots, J_k , and let ζ_1, \dots, ζ_r be slopes on T_1, \dots, T_r , respectively.*

Suppose that G is either $\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}$ or $\text{PSL}(2; \mathbb{C})$, and that

$$\rho : \pi_1(M) \rightarrow G$$

is a representation vanishing on the slopes ζ_i , and that $\widehat{\rho}_i : \pi_1(\widehat{J}_i) \rightarrow G$ are the induced representations, where \widehat{J}_i is the Dehn filling of J_i along slopes adjacent to its boundary, with the induced orientations. Then:

$$\text{vol}_G(M, \rho) = \text{vol}_G(\widehat{J}_1, \widehat{\rho}_1) + \text{vol}_G(\widehat{J}_2, \widehat{\rho}_2) + \dots + \text{vol}_G(\widehat{J}_k, \widehat{\rho}_k).$$

The following simple lemma suggests that we should focus on those JSJ pieces whose groups have non-elementary images under ρ .

Lemma 3.2 ([DLW, Lemma 3.6]). *Suppose that G is either $\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}$ or $\text{PSL}(2; \mathbb{C})$ and that M is a closed oriented 3-manifold. If $\rho : \pi_1 M \rightarrow G$ has image either infinite cyclic or finite, then $\text{vol}_G(M, \rho) = 0$.*

The existence of a class inversion for the target group has played an important role in [DLW] for constructing virtual representation of mixed 3-manifold groups. Here we quote the following definition. An intimately related notion called CI completion is introduced and studied in this paper when we prove Theorem 1.8 (see Subsection 5.2).

Definition 3.3 ([DLW, Definition 5.1]). Let \mathcal{G} be an arbitrary group and $\{[A_i]\}_{i \in I}$ be a collection of conjugacy classes of abelian subgroups. By a *class inversion* with respect to $\{[A_i]\}_{i \in I}$, we mean an outer automorphism $[\nu] \in \text{Out}(\mathcal{G})$, such that for any representative abelian subgroup A_i of each $[A_i]$, there is a representative automorphism $\nu_{A_i} : \mathcal{G} \rightarrow \mathcal{G}$ of $[\nu]$ that preserves A_i , taking every $a \in A_i$ to its inverse. We say \mathcal{G} is *class invertible* with respect to $\{[A_i]\}_{i \in I}$, if there exists a class inversion. We often ambiguously call any collection of representative abelian subgroups $\{A_i\}_{i \in I}$ a class invertible collection, and call any representative automorphism ν a class inversion.

Now we state the following fundamental construction about virtual representation extensions. It uses works of Przytycki–Wise [PW1, PW2, PW3] (and [Wi, HW]) and Rubinstein–Wang [RW] (see also [Li]) to understand virtual properties of 3-manifolds with nontrivial geometric decomposition.

Theorem 3.4 ([DLW, Theorem 5.2]). *Let \mathcal{G} be a group, and M be an irreducible orientable closed 3-manifold with non-trivial JSJ decomposition. For a fixed JSJ piece $J_0 \subset M$, suppose a representation*

$$\rho_0 : \pi_1(J_0) \rightarrow \mathcal{G}$$

satisfies the following:

- *for every boundary torus $T \subset \partial J_0$, ρ_0 has nontrivial kernel restricted to $\pi_1(T)$; and*
- *for all boundary tori $T \subset \partial J_0$, $\rho_0(\pi_1(T))$ form a class invertible collection of abelian subgroups of \mathcal{G} .*

Then there exist a finite regular cover

$$\kappa : \tilde{M} \rightarrow M,$$

and a representation

$$\tilde{\rho} : \pi_1(\tilde{M}) \rightarrow \mathcal{G},$$

satisfying the following:

- *for one or more elevations \tilde{J}_0 of J_0 , the restriction of $\tilde{\rho}$ to $\pi_1(\tilde{J}_0)$ is, up to a class inversion, conjugate to the pull-back $\kappa^*(\rho_0)$; and*
- *for any elevation \tilde{J} other than the above, of any geometric piece J , the restriction of $\tilde{\rho}$ to $\pi_1(\tilde{J})$ is cyclic, possibly trivial.*

3.2. Proof of Theorem 1.6. Now we are ready to prove Theorem 1.6, and here is a sketch of the strategy. Since we can suppose that the manifold has a hyperbolic JSJ piece, Theorem 1.10 gives a virtual representation of the hyperbolic piece with positive Seifert volume. Then with Lemma 3.7, Theorem 3.4 extends the virtual representation to the whole manifold, and the volume of the virtual representation can be calculated by Theorem 3.1 and Lemma 3.2.

By Theorem 1.3 and Theorem 1.10, we may assume that M has non-trivial JSJ decomposition and contains a hyperbolic JSJ piece J in M . Suppose ∂J is a union of tori T_1, \dots, T_k . Let α_i be a slope on T_i , then call $\alpha = \{\alpha_1, \dots, \alpha_k\}$ a slope on ∂M . Denote by $J(\alpha)$ the closed orientable 3-manifold obtained by Dehn filling of k solid tori S_1, \dots, S_k to J along α . We can choose α so that $J(\alpha)$ is a hyperbolic 3-manifold (Theorem 2.1).

Take a closed orientable manifold N of non-vanishing Seifert volume. For example, a circle bundle N with Euler class $e \neq 0$ over a closed surface of Euler characteristic $\chi < 0$ works: In fact, for such N ,

$$\text{SV}(N) = \frac{4\pi^2|\chi|^2}{|e|} > 0.$$

By Theorem 1.10 there is a finite cover $q : Q \rightarrow J(\alpha)$ such that Q dominates N , therefore $\text{SV}(Q) > 0$. Let $S = \cup S_i$, then $S' = q^{-1}(S) \subset Q$ is a union of solid tori and $J' = Q \setminus S'$ is a connected 3-manifold which covers J . Moreover, Q is obtained by Dehn filling S' to J' along α' , where α' is a slope of $\partial J'$ which covers α (i.e. each component of α' is an elevation of a component of α). Fix J' and α' for the moment, we have the following statement.

Lemma 3.5. *Let \tilde{J} be a finite covering of J' and $\tilde{\alpha}$ be the slope of $\partial\tilde{J}$ which covers α' , then $\text{SV}(\tilde{J}(\tilde{\alpha})) > 0$.*

Proof. By the choice of $\tilde{\alpha}$, the covering $\tilde{J} \rightarrow J'$ extends to a branched covering $\tilde{J}(\tilde{\alpha}) \rightarrow J'(\alpha')$ branched over the cores of S' . Since the branched covering is a map of non-zero degree, and $\text{SV}(J'(\alpha')) > 0$, we have $\text{SV}(\tilde{J}(\tilde{\alpha})) > 0$. \square

According to [DLW, Proposition 4.2], there is a finite cover $p : \tilde{M} \rightarrow M$ such that each JSJ piece \tilde{J} of \tilde{M} that covers J factors through J' . In particular, in the notations we have just used, $\text{SV}(\tilde{J}(\tilde{\alpha})) > 0$. To simplify the notations, we still rewrite \tilde{M} , \tilde{J} , $\tilde{\alpha}$ as M , J , α . Since Theorem 1.6 concludes with a virtual property, we need only to prove the following statement.

Theorem 3.6. *Suppose M is a closed orientable 3-manifold with non-trivial JSJ decomposition, and there is a JSJ piece J and a slope α of ∂J such that $\text{SV}(J(\alpha)) > 0$. Then there is a finite cover \tilde{M} of M such that $\text{SV}(\tilde{M}) > 0$.*

We are going to apply Theorem 3.4 to prove Theorem 3.6. So we first need to check that the 3-manifold M and the local representation $\rho : \pi_1(J) \rightarrow G$ (which gives positive representation volume for $J(\alpha)$) in Theorem 3.6 meet the two conditions of Theorem 3.4.

We first write a presentation of $\pi_1(J(\alpha))$ from $\pi_1(J)$ by attaching k relations from Dehn fillings. Let $G = \widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})}$ be the identity component of $\widetilde{\text{IsoSL}_2(\mathbb{R})}$, the isometry group of the Seifert space $\widetilde{\text{SL}_2(\mathbb{R})}$. Then the condition $\text{SV}(J(\alpha)) > 0$ implies that there is a representation $\rho : \pi_1(J) \rightarrow G$ such that for each component T_i of ∂J , $\rho(\pi_1(T_i))$ is a (possibly trivial) cyclic group. Moreover, ρ extends to $\hat{\rho} : \pi_1(J(\alpha)) \rightarrow G$ such that $V_G(J(\alpha), \hat{\rho}) > 0$. So the first condition of Theorem 5.2 of [DLW] is satisfied. The following lemma, which strengthens [DLW, Lemma 6.1 (2)], implies that the second condition of Theorem 5.2 of [DLW] is also satisfied.

Lemma 3.7. *$\widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})}$ is class invertible with respect to all its cyclic subgroups, and a class inversion can be realized by an inner automorphism of $\widetilde{\text{IsoSL}_2(\mathbb{R})}$. The corresponding action on $\widetilde{\text{SL}_2(\mathbb{R})}$ preserves the orientation.*

Proof. There are short exact sequences of groups

$$0 \longrightarrow \mathbb{R} \longrightarrow \widetilde{\text{IsoSL}_2(\mathbb{R})} \xrightarrow{p} \text{Iso } \mathbf{H}^2 \longrightarrow 1$$

and

$$0 \longrightarrow \mathbb{R} \longrightarrow \widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})} \xrightarrow{p} \text{Iso}_+ \mathbf{H}^2 \longrightarrow 1.$$

Recall that there are no orientation reversing isometries in the $\widetilde{\text{SL}_2(\mathbb{R})}$ geometry.

For each element ν in the component of $\widetilde{\text{IsoSL}_2(\mathbb{R})}$ not containing the identity, ν reverses the orientation of \mathbb{R} (the center of $\widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})}$). So $\nu r \nu^{-1} = r^{-1}$ for any $r \in \mathbb{R}$, and $\widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})}$ is class invertible with respect to its center \mathbb{R} , and a class inversion can be realized by an inner automorphism of $\widetilde{\text{IsoSL}_2(\mathbb{R})}$, the corresponding action on $\widetilde{\text{SL}_2(\mathbb{R})}$ preserves the orientation. Actually, this part of the proof is same with the proof of [DLW, Lemma 6.1(ii)].

In the following, we suppose that $\langle \alpha \rangle$ is a cyclic subgroup of $\widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})}$ generated by a non-central element α .

For each non-trivial element a in $\text{Iso}_+ \mathbf{H}^2$, it is straightforward to see that there exists a reflection about a geodesic l_a in \mathbf{H}^2 that conjugates a to its inverse. The l_a can be chosen as: (i) passing through the rotation center when a is elliptic, (ii) perpendicular with the axis of a when a is hyperbolic, (iii) passing through the fixed point when a is parabolic.

By the discussion in the last paragraph and the exact sequences, there exists an element $\nu \in \widetilde{\text{IsoSL}_2(\mathbb{R})}$ lying in the component of $\widetilde{\text{IsoSL}_2(\mathbb{R})}$ not containing the identity, such that $p(\nu)$ is a reflection of \mathbf{H}^2 conjugating $p(\alpha)$ to its inverse, namely, $p(\nu^{-1}\alpha\nu) = p(\alpha^{-1})$. We claim that

$$\nu^{-1}\alpha\nu = \alpha^{-1}.$$

In fact, by the short exact sequences above, we have that $\nu^{-1}a\nu = a^{-1}r$ for some r in the center \mathbb{R} . Since $p(\nu)$ is a reflection, ν^2 is central, so

$$a = \nu^{-2}a\nu^2 = \nu^{-1}(a^{-1}r)\nu = (\nu^{-1}r\nu)(\nu^{-1}a\nu)^{-1} = r^{-1}(a^{-1}r)^{-1} = ar^{-2}.$$

Here we used that fact that ν is a class inversion for $\langle r \rangle$. So r^{-2} is trivial, and r is trivial as the center is torsion free. This verifies the claim. We conclude that ν realizes a class inversion of the cyclic subgroup $\langle a \rangle$ of $\widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})}$.

Suppose we have two elements $\alpha_1, \alpha_2 \in \widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})}$. Then there exist $\nu_1, \nu_2 \in \widetilde{\text{IsoSL}_2(\mathbb{R})}$ such that $\nu_i^{-1}\alpha_i\nu_i = \alpha_i^{-1}$, and $p(\nu_i)$ is a reflection of \mathbf{H}^2 , for $i = 1, 2$. Since any two reflections on \mathbf{H}^2 are conjugate with each other by an element $b \in \text{Iso}_+ \mathbf{H}^2$, we have $p(\beta)p(\nu_1)p(\beta^{-1}) = p(\nu_2)$ for some $\beta \in \widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})}$. Therefore we have $\beta\nu_1\beta^{-1} = \nu_2r$ for some $r \in \mathbb{R}$.

Clearly ν_2r plays the same role as ν_2 did, so ν_1 and ν_2 can be chosen to be conjugate with each other by an element in $\widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})}$. We have verified that $\widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})}$ is class invertible with respect to all its cyclic subgroups, and a class inversion can be realized by an inner automorphism of $\widetilde{\text{IsoSL}_2(\mathbb{R})}$, and the corresponding action on $\widetilde{\text{SL}_2(\mathbb{R})}$ preserves the orientation. \square

Proof of Theorem 3.6. Fix J , α , and $\rho : \pi_1(J) \rightarrow G$ as our previous discussion, and denote them by J_0 , α_0 , and ρ_0 to match the notations of Theorem 3.4. Since $\rho_0 : \pi_1(J_0) \rightarrow G$ meets the two conditions of Theorem 3.4, we can virtually extend ρ_0 to $\tilde{\rho} : \pi_1(\tilde{M}) \rightarrow G$, which satisfies those conditions in the conclusion of Theorem 3.4.

By the additivity principle (Theorem 3.1), we need only to compute the representation volume for each JSJ piece of \tilde{M} , then add the volumes together to compute $V_G(\tilde{M}, \tilde{\rho})$. By Theorem 3.4 and Lemma 3.2, only those elevations \tilde{J}_0 of J_0 such that the restriction of $\tilde{\rho}$ to $\pi_1(\tilde{J}_0)$ is conjugate to the pull-back $\kappa^*(\rho_0)$, up to a class inversion, could contribute to the Seifert representation volume of \tilde{M} .

By Lemma 3.7, the class inversions can be realized by conjugations of orientation preserving isomorphisms of $\widetilde{\text{SL}_2(\mathbb{R})}$, therefore the volumes of all these elevations are positive multiples of $V_G(J_0(\alpha_0), \rho_0) > 0$. So the Seifert representation volume of \tilde{M} with respect to $\tilde{\rho}$ is positive, which implies $\text{SV}(\tilde{M}) > 0$. \square

The completion of the proof of Theorem 3.6 also completes the proof of Theorem 1.6.

Remark 3.8. We can reformulate what we have done in this section by the following proposition:

Proposition 3.9. *Let M be an orientable closed mixed 3-manifold and J_0 be a distinguished hyperbolic JSJ piece of M . Suppose that \hat{J}_0 is a closed hyperbolic Dehn filling of J_0 by sufficiently long boundary slopes.*

- (1) *For any finite cover \hat{J}'_0 of \hat{J}_0 and any representation*

$$\eta : \pi_1(\hat{J}'_0) \longrightarrow \text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})},$$

there exist a finite cover

$$\tilde{M}' \longrightarrow M$$

and a representation

$$\rho : \pi_1(\tilde{M}') \longrightarrow \text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})},$$

with the following properties:

- *For one or more elevations \tilde{J}' of J_0 contained in \tilde{M}' , the covering $\tilde{J}' \rightarrow J_0$ factors through a covering $\tilde{J}' \rightarrow J'_0$, where $J'_0 \subset \hat{J}_0$ denotes the unique elevation of $J_0 \subset \hat{J}_0$. The restriction of ρ to $\pi_1(\tilde{J}')$ is conjugate to either the pull-back $\beta^*(\eta)$ or the pull-back $\beta^*(\nu\eta)$, where ν is a class inversion, and β is the composition of the mentioned maps:*

$$\tilde{J}' \xrightarrow{\text{cov}} J'_0 \xrightarrow{\text{fill}} \hat{J}'_0;$$

- *For any elevation \tilde{J}' other than the above, of any JSJ piece J of M , the restriction of ρ to $\pi_1(\tilde{J}')$ has cyclic image, possibly trivial.*

- (2) $\text{Vol}_{\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}}(\tilde{M}', \rho)$ *is a positive multiple of* $\text{Vol}_{\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}}(\hat{J}'_0, \eta)$.

The first part of Proposition 3.9 is a specialized refined statement of Theorem 3.4; the second part supplies a slot to connect with Theorem 1.10. Therefore, Theorem 1.6 is a consequence of Proposition 3.9 and Theorem 1.10. The stronger result, Theorem 1.8, will follow from an efficient version of this proposition (Theorem 5.1) plus the efficient virtual domination (Theorem 1.11).

4. EFFICIENT VIRTUAL DOMINATION BY HYPERBOLIC 3-MANIFOLDS

In this section, we employ the work of A. Gaifullin [Ga] to derive Theorem 1.11 from Theorem 1.10. We quote the statement below for convenience.

Theorem (1.11). *For any closed oriented hyperbolic 3-manifold M , there exists a positive constant $c(M)$ such that the following statement holds. For any closed oriented 3-manifold N and any $\epsilon > 0$, there exists a finite cover M' of M which admits a non-zero degree map $f : M' \rightarrow N$, such that*

$$\|M'\| \leq c(M) \cdot |\deg(f)| \cdot (\|N\| + \epsilon).$$

Remark 4.1. In fact, the same statement holds for any closed orientable manifold which virtually dominates all closed orientable manifolds of the same dimension. For dimension 3, all hyperbolic manifolds have such property [Su]. For any arbitrary dimension, manifolds with this property have been discovered by A. Gaifullin [Ga].

The 3-dimension example M_{Π^3} of Gaifullin is not a hyperbolic manifold, but we point out that a constant $c_0 = 24v_8/v_3 \approx 86.64$ is sufficient for this case, where v_8 is the volume of the ideal regular hyperbolic octahedron and v_3 is the volume of the ideal regular hyperbolic tetrahedron.

4.1. URC manifolds. As introduced by A. Gaifullin [Ga], a closed orientable (topological) n -manifold M is said to have the property of *Universal Realisation of Cycles (URC)* if every homology class of $H_n(X; \mathbb{Z})$ of an arbitrary topological space X has a positive integral multiple which can be realized by the fundamental class of a finite cover M' of M , via a map $f : M' \rightarrow X$.

For any arbitrary dimension n , Gaifullin shows that examples of URC n -manifolds can be obtained by taking some 2^n -sheeted cover

$$M_{\Pi^n}$$

of some n -dimensional orbifold Π^n . More precisely, the underlying topology space of Π^n is the *permutahedron*, namely, the polyhedron combinatorially isomorphic to the convex hull of the points $(\sigma(1), \dots, \sigma(n+1))$ of \mathbb{R}^{n+1} , where σ runs over all permutations of $\{1, \dots, n+1\}$. The orbifold structure of Π^n is given so that each codimension-1 face is a reflection wall, so each codimension- k face is the local fixed point set of a \mathbb{Z}_2^k -action. The abelian characteristic cover of Π^n on which $H_1(\Pi^n; \mathbb{Z}_2) \cong \mathbb{Z}_2^n$ acts is the orientable closed n -manifold M_{Π^n} . In particular, M_{Π^n} can be obtained by facet pairing of 2^n permutahedra.

The following quantitative version of Gaifullin's proof [Ga, Section 5] is important for our application. Recall that a (compact) *pseudo n -manifold* is a finite simplicial complex of which each simplex is contained in an n -simplex and each $(n-1)$ -simplex is contained in exactly two n -simplices. Topologically, a pseudo n -manifold is just a manifold away from its codimension-2 skeleton. A *strongly connected orientable* pseudo n -manifold means that, away from the codimension-2 skeleton, the manifold is connected and orientable, or equivalently, that the n -dimensional integral homology is isomorphic to \mathbb{Z} . In particular, the concept of (unsigned) mapping degree can be extended similarly to maps between strongly connected orientable pseudo n -manifolds.

Theorem 4.2 (See [Ga, Proposition 5.3]). *For any strongly connected orientable pseudo n -manifold Z , there exists a finite cover M'_{Π^n} of M_{Π^n} and a non-zero degree map $f_1 : M'_{\Pi^n} \rightarrow Z$, such that*

$$\#\{n\text{-permutahedra of } M'_{\Pi^n}\} = (n+1)! \cdot |\deg(f_1)| \cdot \#\{n\text{-simplices of } Z\}.$$

Remark 4.3. The map f_1 is as asserted by [Ga, Proposition 5.3]. The cover $\widehat{M}_{\Pi^n} = \mathcal{U}_{\Pi^n}/\Gamma_H$ there is rewritten as M'_{Π^n} in our paraphrase. To compare with the statement of [Ga, Proposition 5.3], the index $|W : \Gamma_H|$ there equals the number of permutahedra in M'_{Π^n} here; the notation $|A|$ there stands for the number of n -simplices in the barycentric subdivision of Z , which equals $(n+1)!$ times the number of n -simplices of Z here. For dimension 3, all orientable closed hyperbolic manifold are known to be URC [Su].

4.2. Virtual domination through URC 3-manifolds. We combine the results of [Ga] and [Su] to prove Theorem 1.11. The following lemma allows us to create an efficient virtual realization of the fundamental class of N .

Lemma 4.4. *For any closed oriented n -manifold N and any $\epsilon > 0$, there exists a connected oriented pseudo n -manifold Z and a non-zero degree map $f : Z \rightarrow N$, such that*

$$\#\{n\text{-simplices of } Z\} \leq |\deg(f)| \cdot (\|N\| + \epsilon).$$

Proof. By the definition of the simplicial volume, for any $\epsilon > 0$, there exists a singular cycle

$$\alpha = \sum_{i=1}^k s_i \sigma_i \in Z_n(N, \mathbb{R})$$

such that $[\alpha] = [N] \in H_n(N, \mathbb{R})$ and

$$\sum_{i=1}^k |s_i| < \|N\| + \epsilon.$$

Here s_i are real numbers and σ_i are maps from the standard oriented n -simplex to N .

Since

$$\begin{cases} \sum_{i=1}^k x_i \sigma_i & \in Z_n(N, \mathbb{R}) \\ [\sum_{i=1}^k x_i \sigma_i] & = [N] \in H_n(N, \mathbb{R}) \end{cases}$$

can be expressed as linear equations with integer coefficients, it has a rational solution (r_1, \dots, r_k) close to (s_1, \dots, s_k) such that $r_i \in \mathbb{Q}$ and

$$\sum_{i=1}^k |r_i| < \|N\| + \epsilon.$$

In particular, $[\sum_{i=1}^k r_i \sigma_i] = [N] \in H_n(N, \mathbb{R})$ holds. Here we can suppose that each r_i is $-$ negative, by reversing the orientation of σ_i if necessary.

Let the least common multiple of the denominators of r_i be denoted m , then

$$\beta = m \left(\sum_{i=1}^k r_i \sigma_i \right) = \sum_{i=1}^k (mr_i) \sigma_i \in Z_n(N; \mathbb{Z})$$

is an integer linear combination of σ_i and $[\beta] = m[N] \in H_n(N, \mathbb{R})$.

Here we can think that we have mr_i copies of the standard oriented n -simplex that is mapped as σ_i , $i = 1, 2, \dots, k$. The condition that $\sum_{i=1}^k (mr_i) \sigma_i$ being an n -cycle implies that we can find a pairing of all the $(n-1)$ -dimensional faces of the collection of copies of σ_i s, such that each such pair are mapped to the same singular $(n-1)$ -simplex in N , with opposite orientation.

This pairing instructs us to build an oriented pseudo-manifold Z' (possibly disconnected). It is given by taking $\sum_{i=1}^k mr_i$ copies of the standard oriented n -simplex and pasting them together by the pairing given above. Then the singular n -simplices $\{\sigma_i\}_{i=1}^k$ induces a map $f_0 : Z' \rightarrow N$.

Let $[Z']$ be the homology class in $H_n(Z')$ which is represented by the n -cycle which takes each oriented n -simplex in Z' exactly once. It is easy to see that $f_0([Z']) = [\beta] = m[N]$, so f_0 has mapping degree $\deg(f_0) = m$. Moreover, the number of n -simplices in Z' is just

$$\sum_{i=1}^k (mr_i) = m \left(\sum_{i=1}^k r_i \right) < m(\|N\| + \epsilon) = \deg(f_0) \cdot (\|N\| + \epsilon).$$

If Z' is connected, we are done with the proof. If Z' is disconnected, take the component Z of Z' such that

$$\frac{\deg(f_0|_Z)}{\#\{n\text{-simplices of } Z\}}$$

is not smaller than the corresponding number for all the other components of Z' . Then $f = f_0|_Z$ satisfies the desired condition in this lemma. \square

4.2.1. *Construction of (M', f) .* Let M be a closed orientable hyperbolic 3-manifold, and N be any closed orientable 3-manifold. Given any constant $\epsilon > 0$, denote by

$$p : Z \longrightarrow N$$

a virtual realization of the fundamental class of N by a strongly connected orientable pseudo 3-manifold, as guaranteed by Lemma 4.4. Take a finite cover M'_{Π^3} of Gaifullin's URC 3-manifold M_{Π^3} and an efficient domination map

$$f_1 : M'_{\Pi^3} \longrightarrow Z$$

which come from Theorem 4.2. Take a finite cover \tilde{M} of M and a π_1 -surjectively 2-domination map:

$$f_2 : \tilde{M} \longrightarrow M_{\Pi^3}$$

which comes from Theorem 1.10. Then there exists a unique finite cover M' of M , up to isomorphism of covering spaces, and a unique π_1 -surjective 2-domination map $f'_2 : M' \rightarrow \tilde{M}_{\Pi^3}$ that fit into the following commutative diagram of maps:

$$\begin{array}{ccc} M' & \xrightarrow{f'_2} & M'_{\Pi^3} \\ \downarrow & & \downarrow \\ \tilde{M} & \xrightarrow{f_2} & M_{\Pi^3} \end{array}$$

Indeed, M' is the cover of \tilde{M} that corresponds to the subgroup $(f_{2\#})^{-1}(\pi_1(M'_{\Pi^3}))$ of $\pi_1(\tilde{M})$ (after choosing some auxiliary basepoints). The finite cover M' of M and the composed map:

$$f : M' \xrightarrow{f'_2} \tilde{M}_{\Pi^3} \xrightarrow{f_1} Z \xrightarrow{p} N$$

are the claimed objects of Theorem 1.11.

4.2.2. *Verification.* With the notations above, the commutative diagram above implies:

$$\frac{\|M'\|}{\|\tilde{M}\|} = [M' : \tilde{M}] = [M'_{\Pi^3} : M_{\Pi^3}] = \frac{\#\{\text{permutahedra of } M'_{\Pi^3}\}}{\#\{\text{permutahedra of } M_{\Pi^3}\}}.$$

Observe that there are $2^3 = 8$ permutahedra in Gaifullin's URC 3-manifold M_{Π^3} . On the other hand, by Theorem 4.2 and Lemma 4.4, the construction of M'_{Π^3} and Z yields:

$$\begin{aligned} \#\{\text{permutahedra of } M'_{\Pi^3}\} &= 4! \cdot |\deg(f_1)| \cdot \#\{\text{tetrahedra of } Z\} \\ &< 24 \cdot |\deg(f_1)| \cdot |\deg(p)| \cdot (\|N\| + \epsilon) \\ &= \frac{24}{2} \cdot |\deg(f'_2)| \cdot |\deg(f_1)| \cdot |\deg(p)| \cdot (\|N\| + \epsilon) \\ &= 12 \cdot |\deg(f)| \cdot (\|N\| + \epsilon). \end{aligned}$$

Therefore,

$$\|M'\| < \frac{12 \cdot |\deg(f)| \cdot (\|N\| + \epsilon) \cdot \|\tilde{M}\|}{8} = c_0 \cdot |\deg(f)| \cdot (\|N\| + \epsilon).$$

where the constant c_0 is taken to be:

$$c_0 = \frac{3}{2} \cdot \|\tilde{M}\|.$$

Note that the constant $c_0 > 0$ depends only on the hyperbolic 3-manifold M , because \tilde{M} is constructed by Theorem 1.10 without referring to N or ϵ .

This completes the proof of Theorem 1.11.

4.3. Virtual Seifert volume of closed hyperbolic 3-manifolds. We have mentioned in the introduction that Theorem 1.6 for closed hyperbolic 3-manifolds follows directly from Theorem 1.10. Similarly, Theorem 1.8 for hyperbolic closed 3-manifolds is a corollary of Theorem 1.11.

Corollary 4.5. *For any closed oriented hyperbolic 3-manifold M , the set of values*

$$\left\{ \frac{\text{SV}(M')}{[M' : M]} : M' \text{ any finite cover of } M \right\}$$

is not bounded.

Proof. Take a closed orientable manifold N of non-vanishing Seifert volume and vanishing simplicial volume. For example, a circle bundle N with Euler class $e \neq 0$ over a closed surface of Euler characteristic $\chi < 0$ works.

For every positive integer n , apply Theorem 1.11 with $\epsilon = 1/n$. There exists a finite cover $M_n \rightarrow M$ and a non-zero degree map $f_n : M_n \rightarrow N$ such that

$$\|M\| \cdot [M_n : M] = \|M_n\| \leq c(M) \cdot |\deg(f_n)| \cdot (\|N\| + \frac{1}{n}) = c(M) \cdot |\deg(f_n)| \cdot \frac{1}{n}.$$

So we have

$$[M_n : M] \leq \frac{c(M) \cdot |\deg(f_n)| \cdot \frac{1}{n}}{\|M\|}.$$

Since $\text{SV}(M_n) \geq |\deg(f_n)| \cdot \text{SV}(N)$, we have

$$\frac{\text{SV}(M_n)}{[M_n : M]} \geq \frac{|\deg(f_n)| \cdot \text{SV}(N)}{\frac{c(M) \cdot |\deg(f_n)| \cdot \frac{1}{n}}{\|M\|}} = n \cdot \frac{\|M\| \cdot \text{SV}(N)}{c(M)}.$$

Since $K = \|M\| \cdot \text{SV}(N)/c(M)$ is a positive constant, $\{\text{SV}(M_n)/[M_n : M]\}$ is not a bounded sequence, so we are done. \square

5. POSITIVE SIMPLICIAL VOLUME IMPLIES UNBOUNDED VIRTUAL SEIFERT VOLUME

In this section, we prove Theorem 1.8 following the strategy of the proof of Theorem 1.6 summarized in Remark 3.8. The main body of the proof is the following theorem which produces virtual Seifert representations with controlled volume, (compare Proposition 3.9).

Theorem 5.1. *Let M be an orientable closed mixed 3-manifold and J_0 be a distinguished hyperbolic JSJ piece of M . Suppose that \hat{J}_0 is a closed hyperbolic Dehn filling of J_0 by sufficiently long boundary slopes.*

(1) For any finite cover \widehat{J}_0' of \widehat{J}_0 and any representation

$$\eta : \pi_1(\widehat{J}_0') \longrightarrow \text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})},$$

there exist a finite cover

$$\tilde{M}' \longrightarrow M$$

and a representation

$$\rho : \pi_1(\tilde{M}') \longrightarrow \text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})},$$

with the following properties:

- For one or more elevations \tilde{J}' of J_0 contained in \tilde{M}' , the covering $\tilde{J}' \rightarrow J_0$ factors through a covering $\tilde{J}' \rightarrow J_0'$, where $J_0' \subset \widehat{J}_0'$ denotes the unique elevation of $J_0 \subset \widehat{J}_0$. The restriction of ρ to $\pi_1(\tilde{J}')$ is conjugate to either the pull-back $\beta^*(\eta)$ or the pull-back $\beta^*(\nu\eta)$, where β is the composition of the mentioned maps:

$$\tilde{J}' \xrightarrow{\text{cov}} J_0' \xrightarrow{\text{fill}} \widehat{J}_0';$$

- For any elevation \tilde{J}' other than the above, of any JSJ piece J of M , the restriction of ρ to $\pi_1(\tilde{J}')$ has cyclic image, possibly trivial.
- (2) Furthermore, there exists a positive constant α_0 depending only on M and the Dehn filling $J_0 \rightarrow \widehat{J}_0$, such that for any \widehat{J}_0' and η above, the asserted \tilde{M}' and ρ can be constructed so that the sum of the covering degrees $[\tilde{J}' : J_0]$ over all the elevations \tilde{J}' of the β -pull-back type equals $\alpha_0 \cdot [\tilde{M}' : M]$. Therefore,

$$\frac{\text{Vol}_{\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}}(\tilde{M}', \rho)}{[\tilde{M}' : M]} = \alpha_0 \cdot \frac{\text{Vol}_{\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}}(\widehat{J}_0', \eta)}{[\widehat{J}_0' : \widehat{J}_0]}.$$

The rest of this section is devoted to the proof of Theorem 5.1, before which we derive Theorem 1.8 from Theorem 5.1 and Corollary 4.5.

5.1. Proof of Theorem 1.8. Since we have proved Theorem 1.8 for hyperbolic 3-manifolds (Corollary 4.5), we may assume that M is non-geometric with at least one hyperbolic piece, or in other words, *mixed*. The mixed case is derived from the hyperbolic case and Theorem 5.1.

Take a hyperbolic piece J of M and let \widehat{J} be a closed hyperbolic Dehn filling of J . By Corollary 4.5, there are finite covers $\{\widehat{J}_n'\}$ of \widehat{J} such that

$$\frac{\text{SV}(\widehat{J}_n')}{[\widehat{J}_n' : \widehat{J}]} \geq nK$$

for some constant $K > 0$. Let

$$\eta_n : \pi_1(\widehat{J}_n') \rightarrow \text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}$$

be a representation realizing $\text{SV}(\widehat{J}_n')$.

Granting Theorem 5.1, there exist finite covers \tilde{M}'_n of M and representations:

$$\rho_n : \pi_1(\tilde{M}'_n) \rightarrow \text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})},$$

such that

$$\frac{|\text{Vol}_{\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}}(\tilde{M}'_n; \rho_n)|}{[\tilde{M}'_n : M]} = \alpha_0 \cdot \frac{|\text{Vol}_{\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}}(\widehat{J}_n'; \eta_n)|}{[\widehat{J}_n' : \widehat{J}]} = \alpha_0 \cdot \frac{\text{SV}(\widehat{J}_n')}{[\widehat{J}_n' : \widehat{J}]},$$

where the positive constant α_0 is determined by M and $J_0 \rightarrow \widehat{J}_0$. Therefore,

$$\frac{\mathrm{SV}(\tilde{M}'_n)}{[\tilde{M}'_n : M]} \geq \frac{|\mathrm{Vol}_{\mathrm{Iso}_e \widetilde{\mathrm{SL}_2(\mathbb{R})}}(\tilde{M}'_n; \rho_n)|}{[\tilde{M}'_n : M]} = \alpha_0 \cdot \frac{\mathrm{SV}(\widehat{J}'_n)}{[\widehat{J}'_n : \widehat{J}]} \geq n\alpha_0 K,$$

so the sequence $\{\mathrm{SV}(\tilde{M}'_n)/[\tilde{M}'_n : M]\}$ is unbounded. This completes the proof of Theorem 1.8.

5.2. CI completions of hyperbolic 3-manifolds. The statement of Theorem 5.1 (2) suggests certain relation between the asserted representation $\rho : \pi_1(\tilde{M}') \rightarrow \mathrm{Iso}_e \widetilde{\mathrm{SL}_2(\mathbb{R})}$ and the given representation $\eta : \pi_1(\widehat{J}_0) \rightarrow \mathrm{Iso}_e \widetilde{\mathrm{SL}_2(\mathbb{R})}$. It would certainly hold if ρ factored through the restriction of η to some finite covers of \widehat{J}_0 . However, the latter is a much stronger requirement that exceeds our ability. To overcome this difficulty, we examine the machinery of Theorem 3.4 and observe that ρ does factor through a finite cover of certain CW complex associated with \widehat{J}_0 , which looks like \widehat{J}_0 attached with a number of Klein bottles. In the following, we formalize the idea and introduce *CI completions*, where CI is a breviation for class inversion.

In general, given an arbitrary group with a collection of conjugacy classes of abelian subgroups, it is possible to embed the group into a larger group which possesses a class inversion with respect to the induced collection. For concreteness, we only consider the special case of CI completions for orientable closed hyperbolic 3-manifolds, with respect to a collection of mutually distinct embedded closed geodesics.

5.2.1. Construction of the CI completion. Let V be an orientable closed hyperbolic 3-manifold, and let $\gamma_1, \dots, \gamma_s$ be a collection of mutually distinct embedded closed geodesics of V .

The *CI completion* of V with respect to $\gamma_1, \dots, \gamma_s$ is a pair

$$(W, \sigma_W)$$

where W is a specific CW space equipped with a distinguished embedding $V \rightarrow W$ and $\sigma_W : W \rightarrow W$ is a free involution. The construction is as follows.

Take the product space $V \times \mathbb{Z}$ where \mathbb{Z} is endowed with the discrete topology, and for each γ_i , take a cylinder L_i parametrized as $S^1 \times \mathbb{R}$, where S^1 is identified with the unit circle of the complex plane \mathbb{C} . We regard each closed geodesic γ_i as a map $S^1 \rightarrow V$. Identify the circles $S^1 \times \mathbb{Z}$ of L_i with closed geodesics of $V \times \mathbb{Z}$ by taking any point $(z, n) \in S^1 \times \mathbb{Z}$ to either $(\gamma_i(z), n)$ or $(\gamma_i(\bar{z}), n)$ depending on the parity of n . We agree to use $\gamma_i(z)$ for even n and $\gamma_i(\bar{z})$ for odd n . The resulting space $\widetilde{W}_{\mathbb{Z}}$ is equipped with a covering transformation $\sigma : \widetilde{W}_{\mathbb{Z}} \rightarrow \widetilde{W}_{\mathbb{Z}}$ which takes any point $(x, n) \in V \times \mathbb{Z}$ to $(x, n+1)$, and any point $(z, t) \in L_i$ to $(\bar{z}, t+1)$. The quotient of $\widetilde{W}_{\mathbb{Z}}$ by the action of $\langle \sigma^2 \rangle$ is a space W with a covering transformation σ_W induced by σ .

One may visualize the further quotient space $W / \langle \sigma_W \rangle$ as a 3-manifold V with Klein bottles hanging on the closed geodesics γ_i , one on each. Then W is a double cover of that space into which V lifts, and on which the deck transformation σ_W acts. As a CW space with a free involution, the isomorphism type of (W, σ_W) is independent of the auxiliary parametrizations in the construction, and the isomorphism may further be required to fix the distinguished inclusion of V .

5.2.2. *Properties of CI completions.* We study the relation of CI completions with class inversions and their behavior under finite covers.

Proposition 5.2. *Let V be an orientable closed hyperbolic 3-manifold, and let $\gamma_1, \dots, \gamma_s$ be a collection of mutually distinct embedded closed geodesics of V . Denote by (W, σ_W) the CI completion of V with respect to $\gamma_1, \dots, \gamma_s$.*

- (1) *The outer automorphism of $\pi_1(W)$ induced by σ_W is a class inversion of $\pi_1(W)$ with respect to the collection of conjugacy classes of the maximal cyclic subgroups $\pi_1(\gamma_1), \dots, \pi_1(\gamma_s)$ of $\pi_1(W)$ corresponding to the canonically included free loops;*
- (2) *Suppose that \mathcal{G} is a group which possesses a class inversion $[\nu] \in \text{Out}(\mathcal{G})$ with respect to the conjugacy classes of all the cyclic subgroups. Then for any homomorphism $\eta : \pi_1(V) \rightarrow \mathcal{G}$ then there exists an extension of η to $\pi_1(W)$:*

$$\eta : \pi_1(W) \rightarrow \mathcal{G}.$$

Moreover, for any representative automorphisms $\sigma_{W\sharp}, \nu$ of the outer automorphisms accordingly, the image $\eta\sigma_{W\sharp}(\pi_1(V))$ is conjugate to $\nu\eta(\pi_1(V))$ in \mathcal{G} .

- (3) *Suppose that $\kappa : V' \rightarrow V$ is a covering map of finite degree. Denote by $(W', \sigma_{W'})$ the CI completion of V' with respect to all the elevations in V' of $\gamma_1, \dots, \gamma_s$. Then there exists an extension of κ :*

$$\kappa : W' \rightarrow W$$

which is a covering map equivariant under the action of $\sigma_{W'}$ and σ_W . In particular, the covering degree is preserved under the extension.

Proof. Recall that W is topologically the union of V , $\sigma_W(V)$, and annuli A_i , $\sigma_W(A_i)$. Each annulus A_i has its boundary attached to $V \sqcup \sigma_W(V)$ in such a way that $\gamma_i \subset V$ can be freely homotoped to the orientation-reversal of $\sigma_W(\gamma_i) \subset \sigma_W(V)$ through A_i , and the annuli $\sigma_W(A_i)$ make the homotopy as well.

Statement (1) is now obvious from the above description.

Statement (2) can also be seen topologically. To this end, let X be a CW model for the Eilenberg–MacLane CW space $K(\mathcal{G}, 1)$. Uniquely up to free homotopy, the outer automorphism $[\nu]$ can be realized by a map $R : X \rightarrow X$, and the homomorphism η can be realized as a map $f : V \rightarrow X$. With respect to the inclusion $V \rightarrow W$, we define a map $F : W \rightarrow X$, which extends f , as follows. First define the restriction of F to V and $\sigma_W(V)$ are f and Rf , respectively. Since ν is a class inversion, each $f\gamma_i$ is freely homotopic to the orientation-reversal of $Rf\gamma_i$, as a map $S^1 \rightarrow X$, so the homotopy defines maps $F| : A_i \rightarrow X$ and $F| : \sigma_W(A_i) \rightarrow X$. The resulting map $F : W \rightarrow X$ extends $f : V \rightarrow X$, so on the level of fundamental groups, it gives rise to the claimed extension of $\eta : \pi_1(V) \rightarrow \mathcal{G}$ over $\pi_1(W)$.

Statement (3) follows from construction on further quotient spaces. Observe that the quotient space $W / \langle \sigma_W \rangle$, rewritten as \bar{W} , is topologically the union of V and Klein bottles B_i , where B_i are projected from A_i . Then any finite covering map $V' \rightarrow V$ gives rise to a covering map of the same degree $\bar{W}' \rightarrow \bar{W}$. The covering of Klein bottles are induced by the coverings of $\gamma_i \subset \bar{W}$ by their elevations. In fact, the covering $\bar{W}' \rightarrow \bar{W}$ is unique up to homotopy. The covering $\bar{W}' \rightarrow \bar{W}$ induces two equivariant covering maps $W' \rightarrow W$, differing by deck transformation. The one that respects the distinguished inclusions is as claimed. \square

5.3. Virtual representations through CI completions. With our gadgets of CI completions, we invoke Theorem 3.4 to derive the asserted virtual representations of Theorem 5.1.

5.3.1. Construction for the basic level. Let M be an orientable closed mixed 3-manifold and J_0 be a distinguished hyperbolic JSJ piece of M . Suppose that \widehat{J}_0 is a closed hyperbolic Dehn filling of J_0 by sufficiently long boundary slopes, which are denoted by $\gamma_1, \dots, \gamma_s$. Let

$$(W, \sigma_W)$$

be the CI completion of \widehat{J}_0 with respect to $\gamma_1, \dots, \gamma_s$, (see Subsection 5.2.1). Since $\pi_1(W)$ is class invertible with respect to the conjugacy classes of subgroups $\pi_1(\gamma_i)$ (Proposition 5.2 (1)), Theorem 3.4 can be applied with the target group $\pi_1(W)$ and the initial homomorphism:

$$\pi_1(J_0) \longrightarrow \pi_1(W)$$

induced by the composition of the Dehn filling inclusion $J_0 \subset \widehat{J}_0$ and the canonical inclusion $\widehat{J}_0 \subset W$. The output is a finite cover \tilde{M} of M together with a homomorphism:

$$\phi : \pi_1(\tilde{M}) \longrightarrow \pi_1(W),$$

with described restrictions to its JSJ pieces. Since the CI completion W is an Eilenberg–MacLane space $K(\pi_1(W), 1)$, it would be convenient to realize ϕ as a map

$$f : \tilde{M} \longrightarrow W,$$

which is unique up to homotopy.

Suppose for the moment that we are provided with a representation:

$$\eta_0 : \pi_1(\widehat{J}_0) \longrightarrow \widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})},$$

rather than a virtual representation. By Proposition 5.2 (2) and Lemma 3.7, there is an extension over $\pi_1(W)$, (which is still denoted by η_0 , regarding the original one as restriction,) so that the composition:

$$\tilde{\rho} : \pi_1(\tilde{M}) \xrightarrow{\phi} \pi_1(W) \xrightarrow{\eta_0} \widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})}$$

gives rise to a virtual extension of the representation

$$\rho_0 : \pi_1(J_0) \longrightarrow \pi_1(\widehat{J}_0) \xrightarrow{\eta_0|} \widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})}.$$

At this basic level, the virtual extension is nothing but a finer version of Theorem 3.4 for the special case of Seifert representations about mixed 3-manifolds. It exhibits a factorization of $\tilde{\rho}$ through the CI completion $\pi_1(W)$. However, Proposition 5.2 (3) allows us to promote the above construction to deal with virtual representations of $\pi_1(\tilde{J}_0)$.

5.3.2. Construction of (\tilde{M}', ρ) . Now suppose as of Theorem 5.1 that \widehat{J}_0 is a finite cover of \widehat{J}_0 , and

$$\eta : \pi_1(\widehat{J}_0') \longrightarrow \widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})}$$

is a Seifert representation of $\pi_1(\widehat{J}_0')$. Denote by

$$(W', \sigma_{W'})$$

the CI completion of \widehat{J}_0' with respect to all the elevations of $\gamma_1, \dots, \gamma_s$. By Proposition 5.2 (3), there exists a finite covering map

$$\kappa : W' \longrightarrow W$$

which respects the free involutions and the distinguished inclusions. In particular, κ extends the covering $\widehat{J}_0' \rightarrow \widehat{J}_0$ preserving the degree.

Remember that we have obtained a finite cover \tilde{M} and a map $f : \tilde{M} \rightarrow W$ for the basic level. Take any elevation of f with respect to κ , denoted as:

$$f' : \tilde{M}' \longrightarrow W'.$$

This means that the following diagram is commutative up to homotopy:

$$\begin{array}{ccc} \tilde{M}' & \xrightarrow{f'} & W' \\ \downarrow & & \downarrow \kappa \\ \tilde{M} & \xrightarrow{f} & W \end{array}$$

and $\tilde{M}' \rightarrow \tilde{M}$ is the covering of \tilde{M} which is minimal in the sense that it admits no intermediate covering of this property. (More concretely, one may replace W with the mapping cylinder $Y_f \simeq W$, and turn the map f into an inclusion $\tilde{M} \rightarrow Y_f$, then any elevation $\tilde{M}' \rightarrow Y_f'$ of \tilde{M} in the corresponding finite cover $Y_f' \simeq W'$ gives rise to some $f' : \tilde{M}' \rightarrow Y_f' \rightarrow W'$ up to homotopy.) Since W' is a finite cover of W , there are only finitely many such elevations (\tilde{M}', f') up to isomorphism between covering spaces and homotopy. Moreover, the covering degree $[\tilde{M}' : \tilde{M}]$ is bounded by $[W' : W]$. Denote by

$$\phi' : \pi_1(\tilde{M}') \longrightarrow \pi_1(W')$$

the homomorphism (up to conjugation) induced by f' .

Provided with η and ϕ' above, we extend η to be

$$\eta : \pi_1(W') \longrightarrow \widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})}$$

by Proposition 5.2 (1) and (3) and Lemma 3.7. The finite cover

$$\tilde{M}' \longrightarrow M$$

and the representation:

$$\rho : \pi_1(\tilde{M}') \xrightarrow{\phi'} \pi_1(W') \xrightarrow{\eta} \widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})}$$

are the claimed objects in the conclusion of Theorem 5.1.

Homomorphisms which have been presented can be summarized in the following commutative diagram:

$$\begin{array}{ccccc} \pi_1(\widehat{J}_0') & \xrightarrow{\eta|} & \widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})} & & \\ \downarrow \text{incl}_\# & & \downarrow \text{Id} & & \\ \pi_1(\tilde{M}') & \xrightarrow{\phi'} & \pi_1(W') & \xrightarrow{\eta} & \widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})} \\ \downarrow \text{cov}_\# & & \downarrow \kappa_\# & & \\ \pi_1(\tilde{M}) & \xrightarrow{\phi} & \pi_1(W) & & \end{array}$$

The homomorphisms ϕ and ϕ' are realized by maps f and f' respectively. The representation ρ that we have constructed is the composition along the middle row.

We are going to verify Theorem 5.1 (2) in the next three subsections.

5.3.3. Restriction to JSJ pieces. For any elevation $\tilde{J}' \subset \tilde{M}'$ of a JSJ piece $J \subset M$, \tilde{J}' covers a JSJ piece \tilde{J} of \tilde{M} . Since we have constructed ϕ using Theorem 3.4, either the restriction of ϕ to $\pi_1(\tilde{J})$ has cyclic image, or J is the distinguished hyperbolic piece J_0 and the restriction of ϕ to $\pi_1(\tilde{J})$ is one of the following compositions up to conjugation of $\pi_1(W)$:

$$\pi_1(\tilde{J}) \longrightarrow \pi_1(\hat{J}_0) \longrightarrow \pi_1(W)$$

or

$$\pi_1(\tilde{J}) \longrightarrow \pi_1(\hat{J}_0) \longrightarrow \pi_1(W) \xrightarrow{\sigma_W} \pi_1(W).$$

In the cyclic case, the restriction of ϕ' to $\pi_1(\tilde{J}')$ must also have cyclic image as κ_\sharp is injective. Then the restriction of ρ to $\pi_1(\tilde{J}')$ has cyclic image as well. In the other case, the first homomorphism of either composition factors through $\pi_1(J_0)$ via the Dehn filling, possibly after homotopy of f , we may assume that \tilde{J} covers either J_0 or $\sigma_W(J_0)$ under the map f . As f' is an elevation of f with respect to κ , the elevation \tilde{J}' of \tilde{J} covers either the unique elevation J'_0 of J_0 or the unique elevation $\sigma_{W'}(J'_0)$ of $\sigma_W(J_0)$ in W' . Note that η is equivariant up to conjugacy with respect to the class inversions $\sigma_{W'}$ and ν , (Proposition 5.2 and Lemma 3.7). It follows that by taking

$$\beta : \tilde{J}' \longrightarrow J'_0 \longrightarrow \hat{J}_0$$

the composition of the covering and the inclusion, the restriction of ρ to $\pi_1(\tilde{J}')$ is either $\beta^*(\eta)$ or $\beta^*(\nu\eta)$. This verifies Theorem 5.1 (1).

5.3.4. Count of degree. By the consideration about the restriction of ρ to JSJ pieces of \tilde{M}' above, we have seen that a JSJ piece \tilde{J}' gives rise to the β -pull-back type restriction of ρ if and only if \tilde{J}' covers a JSJ piece \tilde{J} of \tilde{M} such that $\phi(\pi_1(\tilde{J}))$ is non-cyclic. The union of all such \tilde{J} in \tilde{M} form a (disconnected) finite cover $\tilde{\mathcal{J}}$ of the distinguished piece $J_0 \subset M$, and the union of all β -pull-back type \tilde{J}' in \tilde{M}' is nothing but the preimage $\tilde{\mathcal{J}}'$ of $\tilde{\mathcal{J}}$ in \tilde{M}' . Therefore, suppose α_0 is the ratio between the total degree of β -pull-back type JSJ pieces of \tilde{M}' over J_0 and the degree of \tilde{M}' :

$$[\tilde{\mathcal{J}}' : J_0] = \alpha_0 \cdot [\tilde{M}' : M],$$

then we observe

$$\alpha_0 = \frac{[\tilde{\mathcal{J}}' : J_0]}{[\tilde{M}' : M]} = \frac{[\tilde{\mathcal{J}}' : \tilde{\mathcal{J}}] \cdot [\tilde{\mathcal{J}} : J_0]}{[\tilde{M}' : \tilde{M}] \cdot [\tilde{M} : M]} = \frac{[\tilde{\mathcal{J}} : J_0]}{[\tilde{M} : M]}.$$

Note that α_0 depends only on M and $J_0 \rightarrow \hat{J}_0$ since \tilde{M} and ϕ are constructed according to them, and α_0 is positive because $\tilde{\mathcal{J}}$ is non-empty by Theorem 3.4.

5.3.5. Count of volume. In a very similar situation as in the proof of Theorem 1.6, to compute the volume of the representation

$$\rho : \pi_1(\tilde{M}') \longrightarrow \widetilde{\text{Iso}_e \text{SL}_2(\mathbb{R})},$$

it suffices to understand the contribution to the representation volume of ρ from the β -pull-back type JSJ pieces \tilde{J}' of \tilde{M}' . Note that the map

$$\beta : \tilde{J}' \xrightarrow{\text{cov}} J'_0 \xrightarrow{\text{fill}} \hat{J}_0$$

factors through a unique hyperbolic Dehn filling \tilde{K}' of \tilde{J}' which covers \hat{J}_0 branching over elevations of the core curves γ_i via a map $\hat{\beta}$:

$$\beta : \tilde{J}' \xrightarrow{\text{fill}} \tilde{K}' \xrightarrow{\hat{\beta}} \hat{J}'_0$$

The restriction of ρ to $\pi_1(\tilde{J}')$ thus factorizes as:

$$\pi_1(\tilde{J}') \xrightarrow{\text{fill}} \pi_1(\tilde{K}') \xrightarrow{\hat{\rho}} \text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})},$$

where $\hat{\rho}$ equals the $\hat{\beta}$ -pull-back of η or $\nu\eta$. Note that the class inversion ν of $\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}$ is realized by the conjugation of an orientation-preserving isomorphism of $\text{SL}_2(\mathbb{R})$, so

$$\text{Vol}_{\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}}(\hat{J}'_0; \eta) = \text{Vol}_{\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}}(\hat{J}'_0; \nu\eta).$$

It follows from the additivity principle (Theorem 3.1) that the contribution to the representation volume of ρ from the piece \tilde{J}' equals $\text{Vol}_{\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}}(\tilde{K}'; \hat{\rho})$ and

$$\begin{aligned} \text{Vol}_{\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}}(\tilde{K}'; \hat{\rho}) &= |\deg(\hat{\beta})| \cdot \text{Vol}_{\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}}(\hat{J}'_0; \eta) \\ &= \frac{[\tilde{J}' : J_0]}{[\hat{J}'_0 : \hat{J}_0]} \cdot \text{Vol}_{\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}}(\hat{J}'_0; \eta). \end{aligned}$$

On the other hand, the contribution from any cyclic type JSJ piece \tilde{J}' of \tilde{M}' is always zero by Lemma 3.2. Take the summation of contribution from all JSJ pieces, using the formula of α_0 in the degree count:

$$\begin{aligned} \text{Vol}_{\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}}(\tilde{M}; \rho) &= \sum_{\tilde{J}' \in \tilde{\mathcal{J}}'} \text{Vol}_{\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}}(\tilde{K}'; \hat{\rho}) \\ &= \sum_{\tilde{J}' \in \tilde{\mathcal{J}}'} \frac{[\tilde{J}' : J_0]}{[\hat{J}'_0 : \hat{J}_0]} \cdot \text{Vol}_{\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}}(\hat{J}'_0; \eta) \\ &= \frac{[\tilde{\mathcal{J}}' : J_0]}{[\hat{J}'_0 : \hat{J}_0]} \cdot \text{Vol}_{\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}}(\hat{J}'_0; \eta) \\ &= \alpha_0 \cdot \frac{[\tilde{M}' : M]}{[\hat{J}'_0 : \hat{J}_0]} \cdot \text{Vol}_{\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}}(\hat{J}'_0; \eta). \end{aligned}$$

or equivalently,

$$\frac{\text{Vol}_{\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}}(\tilde{M}'; \rho)}{[\tilde{M}' : M]} = \alpha_0 \cdot \frac{\text{Vol}_{\text{Iso}_e \widetilde{\text{SL}_2(\mathbb{R})}}(\hat{J}'_0; \eta)}{[\hat{J}'_0 : \hat{J}_0]}.$$

This completes the proof of Theorem 5.1 (2), therefore the proof of Theorem 5.1.

6. ON COVERING INVARIANTS

Although the covering property does not hold for the representation volumes [DLW, Corollary 1.8], we can stabilize them to obtain covering invariants in the following way.

Definition 6.1. For any closed orientable 3-manifold N , define the *covering Seifert volume* of N to be

$$\text{CSV}(N) = \lim_{\leftarrow \tilde{N}} \frac{\text{SV}(\tilde{N})}{[\tilde{N} : N]},$$

valued in $[0, +\infty]$, where \tilde{N} runs over all the finite covers of N . Note that the limit exists because $\text{SV}(\tilde{N}) / [\tilde{N} : N]$ is non-decreasing under passage to finite covers. Similarly one can define the *covering hyperbolic volume* $\text{CHV}(M)$.

Proposition 6.2. *If CSV, or CHV, is valued on $[0, +\infty)$ for a class \mathcal{C} of closed orientable 3-manifolds, then it satisfies both domination property and covering property for \mathcal{C} .*

Proof. We verify the statement for CSV, the argument for CHV completely similar.

To verify the domination property, let $f : M \rightarrow N$ be any map of non-zero degree between $M, N \in \mathcal{C}$. By definition, for any $\epsilon > 0$, there is a finite cover \tilde{N} of N such that

$$\frac{\text{SV}(\tilde{N})}{[\tilde{N} : N]} > \text{CSV}(N) - \epsilon.$$

We have the following commutative diagram:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & \tilde{N} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

for the pull-back cover \tilde{M} of M via f , which has degree at most $[\tilde{N} : N]$. Then we have $[\tilde{M} : M] \cdot |\deg(f)| = [\tilde{N} : N] \cdot |\deg(\tilde{f})|$, and $|\deg(f)| \geq |\deg(\tilde{f})|$, and $\text{SV}(\tilde{M}) \geq |\deg(\tilde{f})| \cdot \text{SV}(\tilde{N})$. It follows that

$$\frac{\text{SV}(\tilde{M})}{[\tilde{M} : M]} = \frac{\text{SV}(\tilde{M}) \cdot |\deg(f)|}{[\tilde{N} : N] \cdot |\deg(\tilde{f})|} \geq \frac{|\deg(f)| \cdot \text{SV}(\tilde{N})}{[\tilde{N} : N]} \geq |\deg(f)| \cdot (\text{CSV}(N) - \epsilon).$$

Taking the limit over all \tilde{M} and $\epsilon \rightarrow 0+$, we have

$$\text{CSV}(M) \geq |\deg(f)| \cdot \text{CSV}(N).$$

To verify the covering property, suppose that $f : M \rightarrow N$ is a covering map, so $\deg(f)$ equals $[M : N]$. Then any finite cover \tilde{M} of M is also a finite cover of N . By definition we have

$$\frac{\text{SV}(\tilde{M})}{[\tilde{M} : M]} = [M : N] \cdot \frac{\text{SV}(\tilde{M})}{[\tilde{M} : N]} \leq [M : N] \cdot \text{CSV}(N) = |\deg(f)| \cdot \text{CSV}(N).$$

Taking the limit over all \tilde{M} , we have $\text{CSV}(M) \leq |\deg(f)| \cdot \text{CSV}(N)$. So indeed we have

$$\text{CSV}(M) = |\deg(f)| \cdot \text{CSV}(N),$$

where the other direction follows from the domination property. \square

We post some further problems, updating those of [DLW, Section 8].

Problem 6.3. Does $\text{CSV}(M)$ exist in $(0, +\infty)$ for every closed orientable non-geometric graph manifold M ?

A positive answer would provide a nowhere vanishing invariant with the covering property in the class of closed orientable non-geometric graph manifolds. Finding such an invariant was suggested by Thurston ([Ki, Problem 3.16]). See [LW, Ne, WW] for some attempts motivated by showing the uniqueness of covering degree between graph manifolds. The uniqueness is confirmed by [YW] using combinatorial method and matrix theory.

Problem 6.4. Determine the possible growth types and asymptotics of the virtual Seifert volume for closed orientable 3-manifolds with positive simplicial volume.

We speak of the growth with respect to towers of finite covers, as the covering degree increases. Theorem 1.8 shows that there are towers with super-linear growth. The estimates of [BG1] imply that the growth must be at most exponential.

Problem 6.5. Is $\text{CHV}(M)$ equal to the simplicial volume for every closed orientable 3-manifold M ?

This quantity is at most the simplicial volume (see Remark 1.9) and we suspect that the equality might be achieved.

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LATP UMR 7353, 39 RUE JOLIOT-CURIE, 13453 MARSEILLE CEDEX 13, FRANCE

E-mail address: pderbez@gmail.com

BELJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH, PEKING UNIVERSITY, BEIJING 100871, CHINA

E-mail address: liuyi@math.pku.edu.cn

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT BERKELEY, CA 94720, USA

E-mail address: hongbins@math.berkeley.edu

DEPARTMENT OF MATHEMATICS, PEKING UNIVERSITY, BEIJING 100871, CHINA

E-mail address: wangsc@math.pku.edu.cn